Methods for Computer Fluid Dynamics aiming Quantum Computing

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In this work, we evaluate Navier-Stokes methods using the **Taylor-Green Vortex** (TGV) test case, aiming at quantum computing feasibility.

The TGV test case is governed by the incompressible Navier-Stokes equations (NSE):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}$ is the velocity field, p is the pressure, ν is the kinematic viscosity, and t is time.

The analytical solution for TGV is [1]:

$$\mathbf{u}_{\text{analytical}} = \begin{pmatrix} A\cos(ax)\sin(by)\sin(cz)e^{-\nu(a^2+b^2+c^2)t} \\ B\sin(ax)\cos(by)\sin(cz)e^{-\nu(a^2+b^2+c^2)t} \\ C\sin(ax)\sin(by)\cos(cz)e^{-\nu(a^2+b^2+c^2)t} \end{pmatrix}, \tag{2}$$

where A, B, C, a, b, and c are constants, and x, y, and z are Cartesian coordinates.

The studied methods are:

1. **Local Temporal Linearization:** Approximates the nonlinear term using a Taylor series expansion around equilibrium $\mathbf{u}_0$ [2]:

$$(\mathbf{u} \cdot \nabla)\mathbf{u} \approx (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \alpha \left[(\mathbf{u}_0 \cdot \nabla)\delta\mathbf{u} + (\delta\mathbf{u} \cdot \nabla)\mathbf{u}_0\right], \tag{3}$$

where $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$ is a small perturbation, and α is a coefficient that calibrates the velocity field at each time step.

2. **SVD Matricial Tensorial Linearization:** Uses Singular Value Decomposition (SVD) to approximate velocity fields in low-rank format, reducing complexity while preserving flow features. The 3D fields are reshaped into 2D matrices, decomposed via SVD, and rebuilt using dominant singular components:

$$\mathbf{u}_{\text{approx}} = \mathbf{U}_{:,1:r} \cdot \mathbf{S}_{1:r,1:r} \cdot \mathbf{V}^T_{1:r,:}, \tag{4}$$

where $\mathbf{U}$, $\mathbf{S}$, and $\mathbf{V}$ are singular vectors and values, and r is the rank. The linearized fields are updated as:

$$\mathbf{u}_{\text{linearized}} = \mathbf{u}_{\text{approx}} - \alpha \mathbf{u}_{\text{approx}} \Delta t, \tag{5}$$

where α is a linearization coefficient and Δt is the time step. This approach aligns with low-rank solvers for Navier-Stokes equations [3].

3. Logarithmic Linearization: This method transforms the velocity fields into a logarithmic space to linearize the non-linear terms. The logarithmic transformation is applied to the absolute values of the velocity components, ensuring positivity and avoiding singularities by adding a small

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constant $\epsilon = 10^{-10}$. The linearization is then performed in the logarithmic space, and the results are transformed back to the original space using the exponential function while preserving the direction of the velocity components. The mathematical formulation is as follows:

$$\mathbf{u}_{\log} = \log(|\mathbf{u}| + \epsilon), \quad \mathbf{u}_{\text{linearized}} = \exp(\mathbf{u}_{\log} - \alpha \mathbf{u}_{\log} \Delta t) \cdot \frac{\mathbf{u}}{|\mathbf{u}|}, \tag{6}$$

where α is a linearization coefficient and Δt is the time step. This approach is particularly useful for handling large gradients and ensuring numerical stability in turbulent flows [4].

Periodic boundary conditions were used, and the spatial domain was discretized on a 64^3 grid with 2π length in each direction. The kinematic viscosity $\nu = 0.01\ m/s$ and time step $\Delta t = 0.001\ s$ ensured stability and vortex capture. The total simulated time was $t = 0.001\ s$.

The numerical solution used the 3th-order Runge-Kutta method for temporal integration and centered finite differences for spatial derivatives. The Poisson equation for pressure was solved using Fourier transforms [5].

The mean squared errors (MSE) and mean absolute errors were calculated:

- **Navier-Stokes**: MSE = 0.049062, Absolute Error = 0.161511
- **Local Temporal Linearization**: MSE = 0.049062, Absolute Error = 0.161511
- **SVD Linearization**: MSE = 0.049001, Absolute Error = 0.161220
- **Logarithmical Linearization**: MSE = 0.049049, Absolute Error = 0.161493

SVD Matricial Tensorial Linearization showed the lowest errors, making it a candidate for simplifying NSE without significant accuracy loss. This method is promising for quantum computing applications, where linearization is crucial [6].

References

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